Qualifying Exam—Numerical ODE 08/2022

Show intermediate results at all steps!

1.(10pt) For ODE $\dot{x} = f(x)$ with f satifying the Lipshcitz condition with bounded 2nd order derivative, prove the global convergence of the Euler scheme:

$$x_{n+1} = x_n + kf(x_n).$$

2.(10pt) The Asymptotic stability of a numerical method can be studied through the Harmonic oscillator

$$\frac{dz}{dt} = Az, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad z = (q, v)^T.$$

Derive the stability condition for the Verlet algorithm:

$$\begin{aligned} v^{n+1/2} &= v^n - \frac{k}{2}\omega^2 q^n, \\ q^{n+1} &= q^n + k v^{n+1/2}, \\ v^{n+1} &= v^{n+1/2} - \frac{k}{2}\omega^2 q^{n+1}. \end{aligned}$$

3.(10pt) For the homogeneous ODE $\dot{u}(t) = f(u)$, show that the Local Truncation Error of the following midpoint method is of order $O(k^3)$:

$$U^{n+1} = U^n + kf\left(U^n + \frac{1}{2}kf(U^n)\right).$$

4.(10pt) Consider the Hamiltonian dynamics:

$$\frac{dq}{dt} = \nabla_p H(q,p), \quad \frac{dp}{dt} = -\nabla_q H(q,p).$$

Show that the implicit Euler-B scheme is symplectic

$$q^{n+1} = q^n + \Delta t \nabla_p H(q^n, p^{n+1}), \quad p^{n+1} = p^n - \Delta t \nabla_q H(q^n, p^{n+1}).$$

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5.(10pt) Consider the linear equation

$$\frac{dz}{dt} = Az, \quad \text{where} \quad A = \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right], \quad z = (q, p)^T.$$

Derive the phase shift of the explicit Euler scheme.

6.(10pt) Show that the implicit mid-point method

$$z^{n+1} = z^n + \Delta t J \nabla_z H \left(\frac{z^{n+1} + z^n}{2} \right), \quad z = (q, p)^T$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, is time reversible for solving Hamiltonian dynamics in Problem 4.